Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

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"An Analytical Study of Some Ordinary Differential Equations with Solutions Using Mathematica and Their Applications"

Pre-Synopsis submitted
For the confirmation of the Ph.D. Topic

In the subject
Mathematics faculty of **Science and Technology**

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Ph.D. Synopsis

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1) Introduction:

Ordinary differential equations play a crucial role in various fields of science and engineering, which offers mathematical modelling for dynamic systems and processes. This research study focuses on an analytical study of ordinary differential equations, emphasizing exact solutions using Mathematica and exploring their practical applications in real-world phenomenon. The analytical approach involves solving linear, non-linear, and systems of ODEs, where exact solutions provide deep insights into system behaviour. However, closed-form solutions are not always feasible, particularly for complex equations.

Thus, computational tools like Mathematica become indispensable, offering symbolic, numerical, and graphical solutions with precision and efficiency. The solutions are analysed for stability, equilibrium points, chaos, and qualitative behaviour, enhancing comprehension of the underlying physical systems modelled by these equations. The study also highlights various applications of ordinary differential equations across physics, economics, biology, and engineering, demonstrating their versatility in modelling natural phenomena, optimizing systems, and predicting outcomes. The study emphasizes three critical aspects: stability, chaos, and bifurcation. Stability analysis helps determine whether small perturbations in initial conditions lead to negligible or significant deviations over time. In contrast, chaos, a hallmark of nonlinear dynamics, refers to systems whose outcomes are highly sensitive to initial conditions. Understanding these phenomena is essential in fields such as physics, biology, and finance, where predicting system behaviour can lead to more effective decision-making. Bifurcation theory explores how small changes in parameters can cause a sudden qualitative change in system behaviour. This is crucial in contexts, where systems transition from stability to instability or from periodic to chaotic behaviour. The study will leverage Mathematica to visualize phase portraits, bifurcation diagrams, and trajectories, facilitating a deeper understanding of the underlying dynamics. By providing a comprehensive analysis of ordinary differential equations and their solutions, this research aims to bridge the gap between theoretical insights and practical applications.

2) Origin of the Research Problem:

Ordinary differential equations arisen from the study of some physical phenomenon such as motion and fluid dynamics. Early mathematicians like Newton, Euler and Leibniz developed ordinary differential equations to model these natural laws. The problem can emerge from an interest in understanding the properties and behaviour of ordinary differential equations such as their stability, existence, uniqueness of solutions. Ordinary differential equations are very important in modelling real-world situations in physics, biology, economics and engineering. Population growth model, Financial model, chemical reaction rate are some well-known models prepared for modelling real world systems in physics, biology, economics and engineering. Analytical solutions to certain ODEs lead to the discovery of special functions like Bessel Functions for vibrating membranes, Legendre Polynomials in spherical harmonics, Hermite Polynomials in quantum mechanics. The research aims to demonstrate how analytical techniques can be enhanced using Mathematica, bridging the gap between theoretical solutions and practical applications. Mathematica is widely used in academia, making it an ideal platform to teach and explore ODEs interactively. The problem focuses on developing a systematic approach to solving and analysing ordinary differential equations using Mathematica, emphasizing analytical solutions and their applications across various fields such as physics, engineering, and economics. Thus, we continue studying ordinary differential equations of some special kind using some analytical techniques and Mathematica.

We know, Abel equation of first kind of the form is given by $y'(x) = a(x) + b(x)y(x) + c(x)y(x)^2 + d(x)y(x)^3 \dots (1)$ and Abel equation of second kind is $y'(x) = \frac{a(x)+b(x)y(x)+c(x)y(x)^2+d(x)y(x)^3}{e(x)+f(x)y(x)} \dots (2)$.

In our study we try to get the solution of first order differential equation of the form $y'(x) = a_0(x) + a_1(x)y(x) + a_2(x)y(x)^2 + a_3(x)y(x)^3 + \cdots + a_{n-1}(x)y(x)^{n-1} + a_n(x)y(x)^n \dots (3)$ We also analyse the solution obtained using Mathematica DSolve function and plotting the graph with varying the power 'n' and coefficient functions $a_i(x)'s$. Unlike in paper [3], we define a new kind of first order differential equation of the following form $y' + P(x)y + Q(x)e^{(2-\alpha)\int P(x)dx}y^2 + R(x)e^{(3-\alpha)\int P(x)dx}y^3 = [Q(x) + R(x)]e^{-\alpha\int P(x)dx} \dots (4)$ whose particular solution is given by $y = e^{-\int P(x)dx}$. We can easily prove that given y is a particular solution of eqn. (4) and then we state a theorem which gives general solution of eqn. (4).

3) Review of Literature (Work done previously in the related area):

Ordinary differential equations are the equations containing derivatives of functions with respect to single variable. They play a crucial role in natural phenomenon in various fields like physics, biology, economics and engineering. The textbook like 'Ordinary Differential Equations' by Tenenbaum and Pollard (1985) [1] provide foundational knowledge on methods for solving ordinary differential equations including exact solutions, series solutions and numerical solutions. The existence and uniqueness of solutions of ordinary differential equations is very well explained in the book 'Theory of ordinary differential equations' by Earl A. Coddington, Norman Levinson [2]. Several analytical techniques are used to solve ordinary differential equations including separation of variables, integrating factors, homogenous and non-homogenous equations, Laplace transform.

In the paper named 'New ordinary differential equations and their analytical solutions' [3] author has focussed on solving special kind of Riccati differential equation given by $y' + P(x)y + Q(x)e^{(2-\alpha)\int P(x)dx}y^2 = Q(x)e^{-\alpha\int P(x)dx}$ with proof. Some solution methods for ordinary differential equations in Mathematica function DSolve is presented in the paper entitled 'Methods in Mathematica for Solving Ordinary Differential Equations' [4]. Most significantly the Abel equation of first kind given by the form $y'(x) = a(x) + b(x)y(x) + c(x)y(x)^2 + d(x)y(x)^3$ and Abel equation of second kind $y'(x) = \frac{a(x)+b(x)y(x)+c(x)y(x)^2+d(x)y(x)^3}{e(x)+f(x)y(x)}$ are solved in this paper [4] using Mathematica function DSolve as shown in the following image.

```
In[13]:= DSolve[y'[x] == -a y[x]^2 - x y[x]^3, y, x]

Out[13]= Solve[-(1/2) a^2 ( 2 ArcTan[(-a-2 x y[x])/(Sqrt[-1-4/a^2] a)] / Sqrt[-1-4/a^2]
          - Log[(-1 - x y[x] (-a - x y[x]))/(x^2 y[x]^2)] ) == C[1] - a^2 Log[x], y[x]]
```

Boyce and DiPrima's 'Elementary Differential Equations and Boundary Value Problems' (2009) [5] provides detailed exploration of the above analytical methods. Whereas Zill's 'Differential Equations with Boundary-Value Problems' (2017) [6] discusses new advanced techniques for solving ordinary differential equations analytically. Ordinary differential equations are widely used in various fields like physics, economics, biology, engineering to model and analyse dynamic systems. Ordinary differential equations model

motion, vibrations, oscillations. Kirchhoff's laws for RLC circuits lead to second-order ordinary differential equations. The logistic growth model describes population growth carrying capacity. Epidemic model like SIR model uses ordinary differential equations to track infection rates over time. An introductory text in 'Nonlinear dynamics and chaos' [7], emphasizing applications in several areas of science, which include vibrations, biological rhythms, insect outbreaks, and genetic control systems. 'Differential Equations, Dynamical Systems, and an Introduction to Chaos' [8], covers the dynamical aspects of ordinary differential equations. It explores the relations between dynamical systems and certain fields outside pure mathematics. 'Differential Equations with Mathematica' [9] integrate new applications from a variety of fields, especially biology, physics, and engineering. The new handbook is also completely compatible with recent versions of Mathematica and is a perfect introduction for Mathematica beginners. The book/CD-ROM package contains built-in commands that lets the user solve problems directly using graphical solutions. It focuses on the most often used features of Mathematica for the beginning Mathematica user. CD-ROM contains all Mathematica inputs from the text, new applications from a variety of fields, including engineering, biology, and physics. All applications were completed using recent versions of Mathematica. In the chapter 'Analytical Solutions of Ordinary Differential Equations' [10], authors discuss the major approaches to obtain analytical solutions of ordinary differential equations. They begin with the solutions of first-order differential equations. Several first-order differential equations can be transformed into two major solution approaches: the separation of variables approach and the exact differential approach. They start with a brief review of both approaches, and then they follow them with two sections on how to reduce other problems to either of these methods. First, they discuss the use of similarity transformations to reduce differential equations to become separable. They show that these transformations cover other well-known approaches, such as homogeneous-type differential equations and isobaric differential equations, as special cases. The next section continues with the search for integrating factors that would transform a given differential equation to become exact. Important special cases of this approach include first-order linear differential equations and the Bernoulli equations. 'Solving Differential Equations in Mathematica' [11] is a Wolfram presentation that provides an overview of Mathematica's framework for solving differential equations. Wolfram Language's differential equation solving functions can be used on many types of differential equations. The functions can automatically choose the appropriate algorithms without the user needing to preprocess them. The paper 'Ordinary Differential Equation and Its Application' [12] go through the

multifaceted applications of ODEs in two distinct but interrelated fields: the domain of deep neural networks and the analysis of economic models.

4) Objectives of Research / Proposed Hypothesis:

The fundamental objectives of this study include

i) To study Riccati and Abel type differential equations and their solutions using analytical methods.

ii) To develop some new special kind of Riccati and Abel type equations varying the parameter or coefficient functions and solving them using analytical methods.

iii) To plot the solutions of such equations using Mathematica with some perturbation in the conditions.

iv) To study in detail stability, chaos and bifurcation of linear systems.

v) To study in detail some results on linearization, Nonlinear systems, Lyapunov functions, Global analysis of the phase plane, Periodic ODE, Stability of A-equations.

vi) To develop new techniques and methodologies for studying ordinary differential equations, and to apply these techniques to specific problems in mathematics and science.

vii) To establish some existence of solutions theorem on ODE along with uniqueness.

5) Methodology to be Adopted:

"An Analytical Study of Some Ordinary Differential Equations with Solutions Using Mathematica and Their Applications" is a work that focuses on the study of ordinary differential equations along with some results and their applications in various areas. The methodology of this study involves the following steps:

i) Definition of Problem: The first step is to define a statement of problem on ordinary differential equations to be analysed.

ii) Literature review: After well defining the problem statement, to conduct a comprehensive review of the existing literature on theory of ordinary differential equations. This involves reading and analysing a variety of published articles, research papers, and books related to the topic.

iii) Analytical Study: To choose a diverse set of some special ordinary differential equations. Apply some standard analytical method or develop some new general method and

compare derived solutions with the solutions already obtained in various mathematical literature.

iv) **Implementation Using Mathematica:** In Mathematica, enter ordinary differential equations as an input using its symbolic computation capabilities and study the solution. Also, generate plots of solutions to provide a visual understanding of the behaviour of solutions under various parameters.

v) **Applications:** Identifying real world problems where ordinary differential equations are used in the domains like physics, biology, economics, or engineering. Using Mathematica to model such kind of scenarios then solving and interpreting results within the context of application.

vi) **Validation and Comparison:** Results are to be compared obtained through the use of Mathematica with the results obtained using analytical methods. And thus, the solution is validated with accuracy, computational efficiency.

vii) **Discussion and Interpretation:** Interpreting the mathematical solutions and their implications in the application context.

viii) **Documentation:** Next is to note methodologies, theorems, equations, codes and results in systematic manner which may include discussion on suitability of Mathematica for solving ordinary differential equations.

ix) **Conclusion and Recommendations:** Research findings and their implications from the current study has to be summarized for the further research. Also, some recommendations for the further extension of research to explore some more complex or coupled ordinary differential equations.

x) **References and Appendices:** Finally, all the relevant references, sources and materials reviewed has to be cited in the systematic manner to acknowledge the previous work. It may also be included Mathematica code snippet, graphs and additional data in appendices.

6) Importance of Study / Society Applications:

This study has vital importance and applications in the society at national as well as international level and has interdisciplinary relevance as stated below:

i) **National Status:** In many universities, research institutes and industries of various counties, research on ordinary differential equations plays an important role in

mathematics, engineering and physical sciences curricula. They also use tools like Mathematica to model real world phenomena which emphasize the practical applications of ordinary differential equations. The industries at national levels such as engineering, finance, medical, technology utilize ordinary differential equations modelling processes like heat transfer, market trends, populations dynamics and MRI scan etc. Differential equations help in predictive modelling for national infrastructure, environmental management, and public health.

ii) **International Status:** With the research of ordinary differential equations there is strong international collaboration between various countries through the journals, conferences and academic partnerships. Organization like International Mathematical Union (IMU), American Mathematical Society (AMS) promotes global standards and facilitates cooperations. Some computational tools like Mathematica, MATLAB have standardized and systematic approaches for solving ordinary differential equations worldwide, leading to the comparable advancements across countries. There are many global challenges and ordinary differential equations are pivotal in such situations like climate modelling, aerospace engineering and epidemiology. For example, modelling pandemic spread. Many international journals frequently publish complex and advance analytical research techniques and computational techniques for ordinary differential equations.

iii) **Interdisciplinary Status:** Ordinary differential equations play pivotal role and have vital applications in various disciplines. Ordinary differential equations underpin disciplines like mechanical, civil, electrical and chemical engineering for designing physical systems. Sciences like physics, chemistry and biology dependant on ordinary differential equations to describe some nature laws such as Newton's law of cooling and motion, bacteria growth and reaction kinetics. In finance and economics, ordinary differential equations are used to model economic growth, pricing dynamics, market trend, and financial derivatives. Some biological models such as population dynamics, drug delivery, and spread of diseases, bridge mathematics and healthcare are very much relied on ordinary differential equations. Models based on ordinary differential equations are also incorporated into machine learning and in artificial intelligence research.

7) Proposed Work Plan / Formulation & Structure of Study:

Year wise plan for the research study and the outcomes to be achieved are as follows:

a) First Year:

i) To understand fundamental concepts relevant to ordinary differential equations.

ii) To identify key papers, books, and authors in the area of differential equations.

iii) To participate in department seminars, workshops, conferences, and guest lectures.

iv) To attend some certified courses on SWAYAM, Coursera or on some any other online platform or to take advanced coursework relevant to the research area.

v) To understand and learn some software's or tools like Mathematica, and MATLAB.

vi) To master LaTeX for writing mathematical documents like research papers, proposals, thesis etc.

vii) To write a research proposal stating the clear objectives, methodology, and expected outcome.

b) Second Year:

i) To study advanced concepts and techniques related to the problem under consideration.

ii) To develop some mathematical model based on the ordinary differential equations.

iii) To study existing and new techniques that can be applied for solving problems relevant to the research area.

iv) To start writing some preliminary results in LaTeX to organize and formalize proofs.

v) To prepare some research papers or at least draft some papers for the publication.

vi) To present findings and conclusions at conferences and seminar to obtain feedback.

vii) To refine the work as per suggestions or feedback.

c) Third Year:

i) To look after the research gap and finalize all proofs and results.

ii) To develop a detailed outline of thesis, including chapters and sections.

iii) To revise the thesis based on feedback and as per suggestions from the research guide.

iv) To submit the thesis to university formatted as per guidelines of university.

v) To prepare a clear and concise presentation and practice for the effective presentation.

vi) To conduct a mock defence with peer team.

8) References:

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[11] Wolfram, Solving Differential Equations in Mathematica.

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